

TA Session 1

Foliations and contact structures in dimension three

Problem 1: twisting. Let ξ be a smooth plane field on \mathbb{R}^3 , and X be a smooth, nonvanishing vector field tangent to ξ .

1. Show that there exist smooth coordinates near $O = (0, 0, 0)$ in which X becomes ∂_x , and ξ becomes

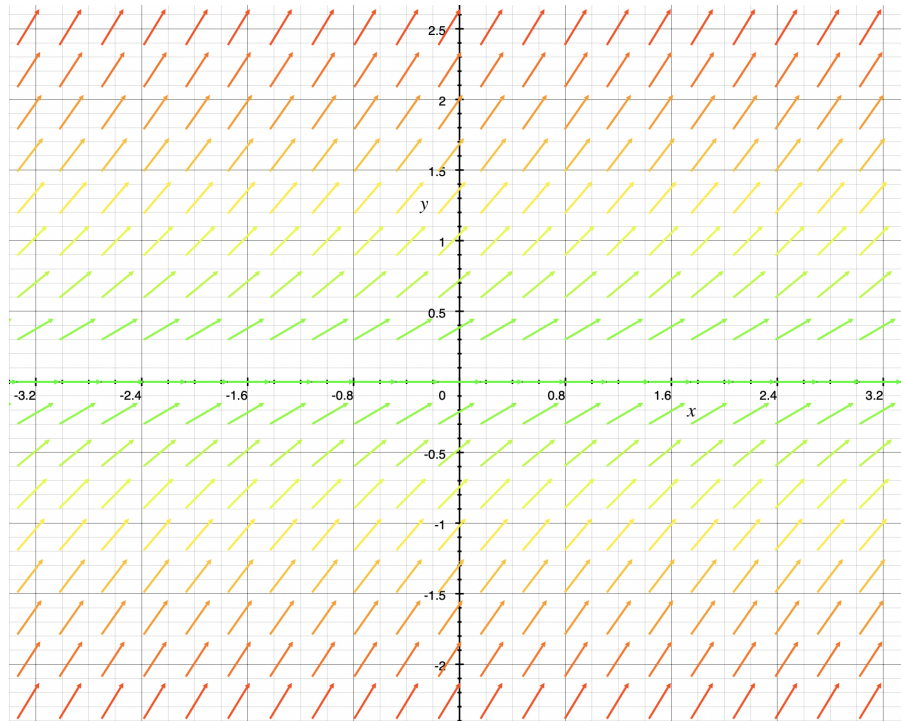
$$\xi = \text{span}(\partial_x, \partial_y + f(x, y, z)\partial_z)$$

for some smooth map f defined near O and satisfying $f(0, y, z) = 0$.

2. Under which condition on f is ξ integrable? contact?
3. If ξ is a positive contact structure, deduce that there are coordinates near O in which f becomes $f(x, y, z) = -x$. Use this to prove Darboux theorem in dimension 3 without using Moser's trick.

Problem 2: a funny vector field. We study the continuous vector field X in \mathbb{R}^2 defined as

$$X(x, y) := \partial_x + \sqrt{|y|} \partial_y.$$



1. Compute the flow lines of X explicitly. What do you notice?
2. Find two topologically crossing flow lines of X .
3. Show that X is tangent to a (unique!) foliation.
4. Consider a flow line of X passing through points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ with $y_1 < 0 < y_2$. Is there an arc transverse to X starting at $p'_1 = (x_1 - \epsilon, y_1)$ and ending at $p'_2 = (x_2 + \epsilon, y_2)$, for a small $\epsilon > 0$?
5. Design a continuous vector field Y on \mathbb{T}^2 satisfying the following properties:
 - Y is tangent to a foliation homeomorphic to a product foliation by circles,
 - There is an embedded annulus $A \subset \mathbb{T}^2$ whose boundaries are tangent to Y with opposite orientations; i.e., A is a dead-end component for Y .

Problem 3: tautness. Let \mathcal{F} be a (co)oriented C^1 foliation on a closed oriented 3-manifold M .

1. Recall that \mathcal{F} is **taut** if for every leaf L of \mathcal{F} , there exists a closed loop transverse to \mathcal{F} crossing L . Show that the following are equivalent:
 - (a) \mathcal{F} is taut,
 - (b) For every $p \in M$, there exists a closed loop transverse to \mathcal{F} passing through p ,
 - (c) There exists a closed 2-form ω which evaluates positively along the leaves of \mathcal{F} ,
 - (d) There exists a smooth volume preserving flow transverse to \mathcal{F} ,
 - (e) \mathcal{F} has no dead-end components.

[Hints: for (b) \implies (c), cover M with tubes foliated by disks, and define ω in these tubes as an exact 1-form. Then use a suitable partition of unity. For (e) \implies (a), consider a leaf L and look at all the points in M that can be accessed from L by an arc transverse to \mathcal{F} . If you try (d) \implies (a), the Poincaré recurrence theorem might be useful. Ask your TA for more hints!]

2. In this question, we don't assume that \mathcal{F} is taut. Show that every noncompact leaf of \mathcal{F} admits a closed transversal. In particular, a foliation without compact leaf is taut.
3. Assume that \mathcal{F} has no torus leaf. Using item (e) above, show that \mathcal{F} is taut.

[Hint: if a leaf of \mathcal{F} is a sphere, then \mathcal{F} is a foliation by spheres on $S^1 \times S^2$ by the Reeb stability theorem. Otherwise, consider a dead-end component N of \mathcal{F} and a transverse vector field to \mathcal{F} , and use the Poincaré–Hopf theorem to compute the Euler characteristic of N , hence of ∂N via $\chi(\partial N) = 2\chi(N)$.]

Problem 4: standard contact structure. On \mathbb{R}^3 , we consider Cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z) . We define

$$\begin{aligned}\alpha_0 &:= dz + xdy, \\ \alpha_1 &:= dz + r^2d\theta,\end{aligned}$$

and $\xi_i := \ker \alpha_i$ for $i \in \{0, 1\}$.

1. Use a Moser argument to show that ξ_0 and ξ_1 are contactomorphic.
2. Compute the characteristic foliation of ξ_1 along the hyperbola $z = \frac{1}{2}xy$. Deduce an explicit contactomorphism between ξ_0 and ξ_1 .

Problem 5: Reeb vector field. Recall that the Reeb vector field of a contact structure ξ for a contact form α is defined by

$$\begin{cases} \alpha(R) = 1 \\ \iota_R d\alpha = 0. \end{cases}$$

1. Show that R preserves ξ , in the sense that its flow ϕ_R^t satisfies $(\phi_R^t)_*\xi = \xi$ for all $t \in \mathbb{R}$.
2. Conversely, show that if Z is a vector field transverse to ξ whose flow preserves ξ , then Z is the Reeb vector field of some contact form α for ξ .
3. Show that there is no closed surface transverse to R .