LOW-DIMENSIONAL SYMPLECTIC DYNAMICS: FROM PERIODIC ORBITS TO BEYOND

PROBLEM SESSION 2

Throughout Problems 1–2, let (Σ, ω) be a closed surface with area form ω and $\int_\Sigma \omega = 1$.

Problem 1. Let $\phi \in \mathrm{Diff}(\Sigma, \omega)$, i.e. $\phi : \Sigma \to \Sigma$ is a diffeomorphism satisfying $\phi^*\omega = \omega$, and let

$$M_\phi = [0,1]_t \times \Sigma / \sim, \qquad (1, p) \sim (0, \phi(p)),$$

be its mapping torus.

(a) Show that M_ϕ admits a 1-form dt and a closed 2-form ω_ϕ such that

$$dt(\partial_t) = 1, \quad \omega_\phi(\partial_t, \cdot) = 0$$

What is $\xi = \ker(dt)$?

(b) A periodic orbit of ϕ is a set of points $\{x_1, \ldots, x_d\} \subset \Sigma$ such that $\phi(x_1) = x_2, \phi(x_2) = x_3, \ldots, \phi(x_d) = x_1$. Show that there is a one-to-one correspondence between closed orbits of ∂_t and periodic orbits of ϕ.

(c) Let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times \Sigma)$ be a time-dependent Hamiltonian (1-periodic in time), and let φ_H^t denote the time-t flow of the Hamiltonian vector field defined by $\omega((X_H)_t, \cdot) = dH_t$. Let $\phi^H = \phi \circ \varphi_H^1$. Show that the diffeomorphism of $[0,1] \times \Sigma$ given by $(t, p) \mapsto (t, (\varphi_H^t)^{-1}(p))$ induces a diffeomorphism $\Psi_H : M_\phi \to M_{\phi^H}$, and that $\Psi_H^*[\omega_{\phi^H}] = [\omega_\phi]$.

Problem 2. Let $\phi \in \mathrm{Diff}(\Sigma, \omega)$. We say that ϕ is *rational* if $[\omega_\phi] \in H^2(M_\phi; \mathbb{Q})$.

(a) Let $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times \Sigma)$ be a Hamiltonian. Show that ϕ is rational if and only if ϕ^H is rational. In other words, rationality is a well-defined property of Hamiltonian isotopy classes.

(b) Using Mayer-Vietoris, show that $H_2(M_\phi; \mathbb{Z}) \cong \mathbb{Z}[\Sigma] \oplus \ker(\phi_* - \mathrm{Id} : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z}))$

(c) Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus, and ω be the restriction of the standard area form on $\mathbb{R}^2$. It is known that every area-preserving diffeomorphism ϕ of $\mathbb{T}^2$ is Hamiltonian isotopic to $Ax + b$, where $A \in \mathrm{SL}_2(\mathbb{Z})$ and $b \in \mathbb{R}^2/\mathbb{Z}^2$. Using part (b), classify rational area-preserving maps on $(\mathbb{T}^2, \omega)$ according to the rank of $A - I$.

Problem 3. Let S^2 be equipped with cylindrical coordinates (θ, z) away from the poles and area form $\omega = \frac{1}{4\pi} d\theta \wedge dz$. For each $k \geq 1$, there is a map

$$f_k : \mathrm{Diff}(S^2, \omega) \to \mathbb{R}$$

defined via the *homogenized link spectral invariants* $\mu_k : C^\infty([0,1] \times S^2) \to \mathbb{R}$. Namely, let $f_k = \mu_k - \mu_1$ by setting

$$f_k(\phi) = \mu_k(H) - \mu_1(H),$$

where $\phi = \varphi_H^1$. The advantage of f_k over μ_k is that it is C^0-continuous and extends to

$$f_k : \mathrm{Homeo}(S^2, \omega) \to \mathbb{R}.$$

Consider the following formal property of μ_k:

- *(Lagrangian control)* Let $H \in C^\infty([0,1] \times S^2)$ be a Hamiltonian, and let $L = L_1 \cup \cdots \cup L_k$ be a *monotone Lagrangian link*, i.e. a union of pairwise disjoint smooth simple closed curves in S^2 such that the connected components of $S^2 \setminus L$ all have equal area. Suppose that, for each $i = 1, \dots, k$, the restriction of H to L_i is a function of t, denoted by c_i. Then

$$\mu_k(H) = \frac{1}{k} \sum_{i=1}^{k} \int_0^1 c_i(t) dt.$$

(a) Let $H \in C^\infty(S^2)$ be an autonomous Hamiltonian depending only on the z-coordinate. How should one choose monotone Lagrangian links in order to compute $f_k(\varphi_H^1)$ using the Lagrangian control property?

(b) Let p_- be the south pole of S^2 whose z-coordinate is -1. Let $H : S^2 \setminus \{p-\} \to \mathbb{R}$ of the form $H(\theta, z) = h(z)$, where $h : (-1, 1] \to \mathbb{R}$ is a smooth function which vanishes for $z \geq -\frac{1}{2}$ and, for $z \leq -\frac{3}{4}$, satisfies the identity $h(z) = \sqrt{\frac{2}{1+z}}$. Show that H induces a well-defined (continuous) flow on S^2 which fixes the point p_- and for $z > -1$,

$$\varphi_H^t(\theta, z) = (\theta + 4\pi h'(z)t, z)$$

(c) Let $T = \varphi_H^1$. Compute $f_k(T)$ using the C^0-continuity of f_k and part (a).

(d) It can be shown that $T \in \mathrm{Hameo}(S^2, \omega)$ and that $\mathrm{Cal}(T) = \frac{1}{2} \int_{-1}^1 h(z)\, dz < \infty$. Use part (c) to verify the asymptotic formula

$$\lim_{k \to \infty} f_k(T) = \mathrm{Cal}(T)$$