LOW-DIMENSIONAL SYMPLECTIC DYNAMICS: FROM PERIODIC ORBITS TO BEYOND

PROBLEM SESSION 1

Problem 1. Consider $\mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$, equipped with the standard symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Let

$$\rho = \frac{1}{2}\sum_{i=1}^{n}(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i})$$

be the radial vector field.

(a) Express ρ in polar coordinates and sketch it in $\mathbb{R}^2$.

(b) Let Y be a hypersurface in $\mathbb{R}^{2n}$, and let $\lambda_0 = \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)$ be the *standard Liouville form* on $\mathbb{R}^{2n}$. Show that if Y is transverse to ρ, then the restriction $\lambda = \lambda_0|_Y$ is a contact form on Y. In this case, Y is called a *star-shaped hypersurface*.

Problem 2. Let $a, b > 0$. Consider the *ellipsoid*

$$E(a,b) = \{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1\}.$$

Let $Y = \partial E(a,b)$. It is easy to check that Y is a star-shaped hypersurface.

(a) Let $\lambda = \lambda_0|_Y$ be the restriction of the Liouville form to Y. Verify that the Reeb vector field of (Y, λ) is

$$R = 2\pi\left(\frac{1}{a}\frac{\partial}{\partial\theta_1} + \frac{1}{b}\frac{\partial}{\partial\theta_2}\right)$$

where θ_j denotes the argument of z_j. Show that if $\frac{a}{b}$ is irrational, then (Y, λ) has exactly two simple Reeb orbits, namely $\gamma_1 = \{z_2 = 0\}, \gamma_2 = \{z_1 = 0\}$.

(b) Draw a suitable visualization of γ_1 and γ_2 from part (a).

Recall that the ECH chain complex is generated by orbit sets. In this case, every generator has the form $\alpha = \gamma_1^{m_1}\gamma_2^{m_2}$ where $m_1, m_2 \in \mathbb{Z}_{\geq 0}$. Since $H_2(Y;\mathbb{Z}) = 0$, for each generator α, there is a unique relative homology class $Z \in H_2(Y, \alpha, \emptyset)$. Thus, we can define a canonical $\mathbb{Z}$-grading using the ECH index:

$$I(\alpha) := I(\alpha, \emptyset, Z) = c_\tau(Z) + Q_\tau(Z) + CZ_\tau(\alpha),$$

Here, τ is a chosen trivialization of ξ along γ_1 and γ_2; $c_\tau(Z) = c_1(\xi|_Z, \tau)$ is the *relative first Chern class*, $Q_\tau(Z)$ is the *relative intersection pairing*, and $CZ_\tau(\alpha)$ is the Conley-Zehnder index term that captures the dynamical information of α.

(c) Choose a trivialization τ over γ_1 and γ_2 as follows. Under the identification $T\mathbb{R}^4 = \mathbb{C}\oplus\mathbb{C}$, the restriction of ξ to γ_1 agrees with the second summand, and the restriction of ξ to γ_2 agrees with the first summand. Explain why this gives a natural choice of trivialization.

(d) Compute $c_\tau(Z)$ as follows: Write $Z = m_1[D_1] + m_2[D_2]$, where $[D_i] \in H_2(Y, \gamma_i, \emptyset)$ for $i = 1, 2$. By additivity, $c_\tau(Z) = m_1 c_{\tau_1}([D_1]) + m_2 c_{\tau_2}([D_2])$, where τ_i is the trivialization of ξ over γ_i. For each i, choose a compact oriented surface D_i with boundary and a map $f_i : D_i \to Y$ representing $[D_i]$; choose a non-vanishing section of $f_i{}^*\xi|_{\partial D_i}$ that is constant with respect to τ_i; extend it generically to a section of $f_i{}^*\xi$ over D_i; and count the signed zeros of this extension.

(e) Compute $Q_\tau(Z)$ as follows: Using notation from part (d), we have

$$Q_\tau(Z) = m_1^2 Q_{\tau_1}([D_1],[D_1]) + m_2^2 Q_{\tau_2}([D_2],[D_2]) + 2m_1m_2 Q_\tau([D_1],[D_2])$$

For each i, choose a properly embedded compact surface D_i in $[-1,1]\times Y$ such that $\partial D_i = \{1\}\times\gamma_i$ and whose projection to Y represents $[D_i]$. Using τ, arrange the boundary of each D_i to approach $\{1\}\times\gamma_i$ in a fixed non-rotating direction. Then compute the last term by slightly perturbing D_1 and D_2 so that they intersect transversely in finitely many interior points and counting these intersections with signs; compute the self-intersection terms similarly using τ-compatible push-offs.

(f) Given that $CZ_\tau^I(\alpha) = \sum_{k=1}^{m_1}\left(2\left\lfloor \frac{ka}{b}\right\rfloor + 1\right) + \sum_{k=1}^{m_2}\left(2\left\lfloor \frac{kb}{a}\right\rfloor + 1\right)$, we obtain, by putting everything together,

$$(\star) \qquad I(\alpha) = 2\left((m_1+1)(m_2+1) - 1 + \sum_{k=1}^{m_1}\left\lfloor \frac{ka}{b}\right\rfloor + \sum_{k=1}^{m_2}\left\lfloor \frac{kb}{a}\right\rfloor\right)$$

In particular, every generator has non-negative even grading. What does this imply about the ECH differential and hence about the ECH of Y?

(g) It turns out that $I(\alpha)$ has a combinatorial interpretation. Let T be the triangle in the plane bounded by the coordinate axes and the line through (m_1, m_2) of slope $-\frac{a}{b}$. Then $\frac{1}{2}I(\alpha)$ equals the number of lattice points in T, including boundary points, minus 1. Use this interpretation to show that $(\star)$ gives a bijection between pairs of nonnegative integers (m_1, m_2) and the set of nonnegative even integers. Conclude that

$$ECH_*(Y,\lambda,0) = \begin{cases} \mathbb{Z}/2, & * = 0, 2, 4, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(h) Let σ be a nonzero ECH class. Recall that the *ECH spectral invariant* $c_\sigma(Y,\lambda)$ is the minimum L such that σ can be represented in the ECH chain complex by a linear combination of generators, each with symplectic action at most L. In our case, for each $k \geq 1$, let σ_k be the class in grading $2k$. Show that $c_{\sigma_k}(Y,\lambda)$ is the smallest L such that the triangle in the plane bounded by the coordinate axes and the line $ax + by = L$ contains $k+1$ lattice points. Use this to verify the ECH Weyl law

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} = \int_Y \lambda\wedge d\lambda$$