Flows and Thurston norm problem sheet
2026 Georgia Topology Conference

All homology/cohomology groups are assumed to have coefficients in $\mathbb{R}$.
Let M be a closed oriented atoroidal 3-manifold supporting a pseudo-Anosov flow φ. The first couple of problems aim to walk you through a proof that the Euler class of φ computes the Thurston norm of the homology classes of surfaces transverse to φ. We give some necessary background.

Definition 1 (Dual norm)**.** The *dual Thurston norm* x^* is defined by identifying H^2 with the dual of H_2 and taking the vector space norm dual to x. That is, for all $u \in H^2(M;\mathbb{R})$,

$$x^*(u) = \sup\{u(\alpha) \mid \alpha \in H_2(M), x(\alpha) \leq 1\}.$$

Definition 2. For each singular orbit γ of φ, define

$$\text{index}(\gamma) = 2 - \#\,\text{prongs}(\gamma),$$

and

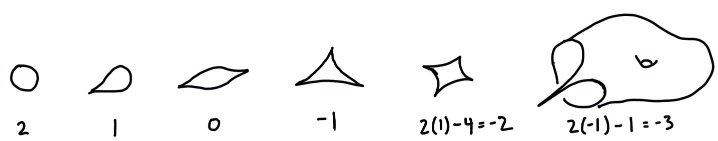
$$\Gamma = \sum_{\gamma} \frac{1}{2}\,\text{index}(\gamma)\cdot[\gamma] \in H_1(M),$$

where the sum is over all singular orbits of φ. Then, for any $u \in H_2(M)$, define

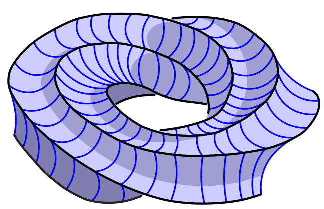
$$e_\varphi(\cdot) = \langle \Gamma, \cdot \rangle$$

where $\langle \cdot, \cdot \rangle$ is algebraic intersection. This linear functional on H_2 is called the *Euler class of* φ.

Definition 3. The index of a compact surface with p cusps on its boundary (see picture) is $2\chi(S) - \#\text{cusps}$.



Definition 4. A *cusped solid torus* is the mapping torus of a diffeomorphism of a closed disc D with possibly cusped boundary (see picture). The *index* of the torus is defined to be the index of D.



Lemma 5. *There is a branched surface in M, which we will call B^s, with the following properties:*

- *The complementary regions of B^s are cusped solid tori with nonpositive index*
- *Each of these cusped solid torus contains a unique singular orbit of φ as its core. The orbit and torus have the same index.*
- *Any essential surface S in M can be homotoped so that the train track $S \cap B^u$ has disc complementary regions with nonpositive index (i.e. there are no smooth disks or monogons).*

(B^s secretly carries the weak stable foliation of φ but we won't need to know this).

1. Using Lemma 5, show that $x^*(e_\varphi) \leq 1$.

   Hint 1: this amounts to showing that, for any norm minimizing surface S we have $|e_\varphi([S])| \leq -\chi(S)$, or $e_\varphi([S]) \geq \chi(S)$.

   Hint 2: Given a train track on a surface S, the sum of the indices of all complementary regions is equal to $2\chi(S)$.

2. Show that if S is a surface transverse to φ, then S realizes the Thurston norm of its homology class, which is given by pairing with $-e_\varphi$. That is,

$$-\chi(S) = x([S]) = -e_\varphi([S]).$$

3. Use the Transverse Surface Theorem to show that $C_2(\varphi)$ is contained in the cone over some face of the Thurston norm ball.

4. Show that the Thurston norm satisfies the triangle inequality.

5. For each of the endperiodic homeomorphisms (f,g,h) shown on the attached sheet, draw the images of the starred segment(s) under a few iterations of the map. See if you can find an *invariant train track* for the map, i.e. a train track that carries its own image (feel free to ask if you don't know what this means).

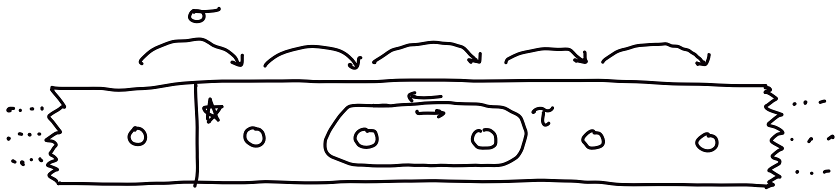
6. (If you are familiar with sutured manifold decomposition)

   (a) Convince yourself that the result of decomposing the link complement shown along the indicated pair of pants is the same as the exterior of the sutured handlebody shown.

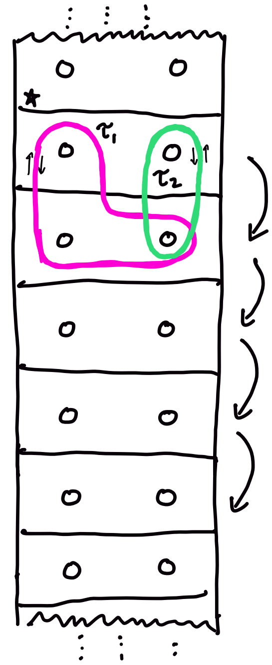
   (b) Decompose along the 2 disks shown (the side facing you is labeled with a minus sign to indicate it becomes part of R_- after decomposing), and convince yourself the result is a product sutured manifold $D^2 \times [0, 1]$.

   (c) Convince yourself that a noncompact leaf of the depth 1 foliation associated to decomposing along the 2 disks is homeomorphic to the surface on which the map h (from an earlier problem) is defined.

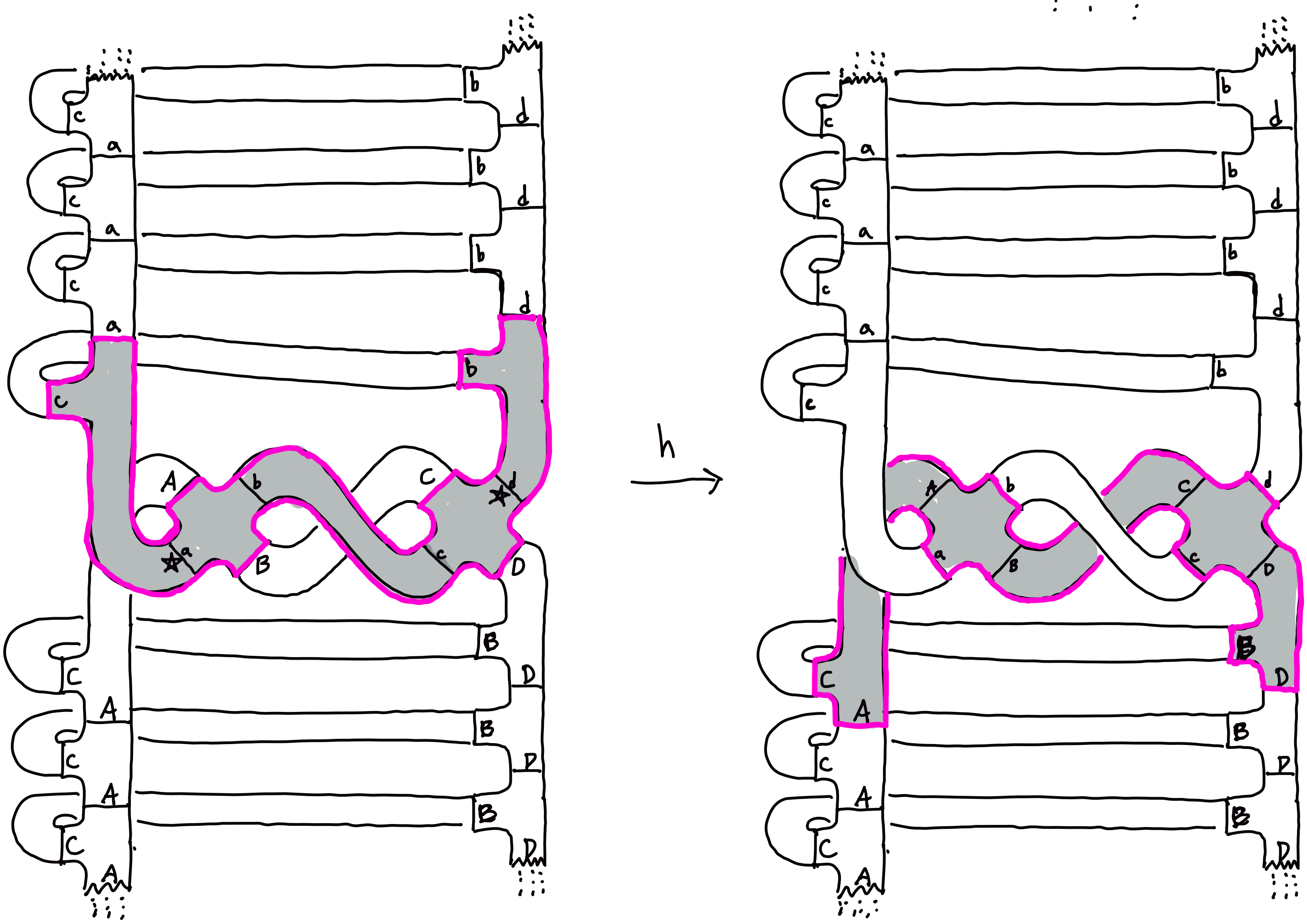
   (d) Convince yourself that the map h is the monodromy of this depth one foliation.

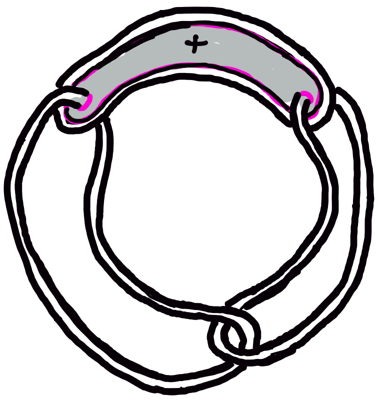


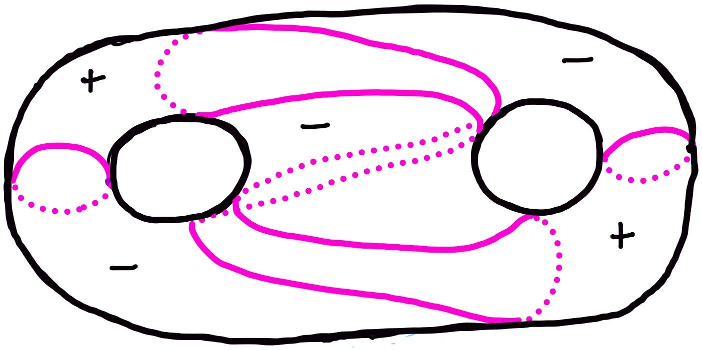
$f = \sigma \circ \tau$

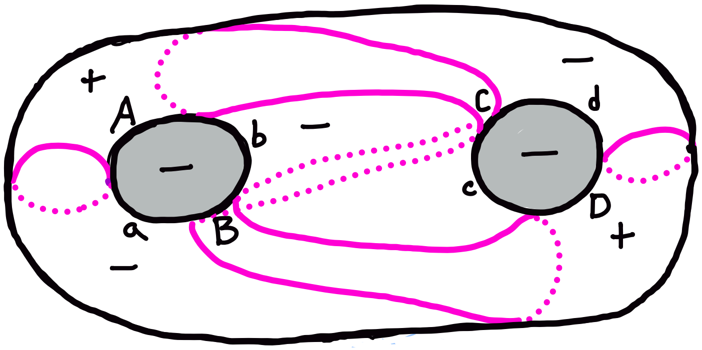


$g = \sigma \circ \tau_2 \circ \tau_1$

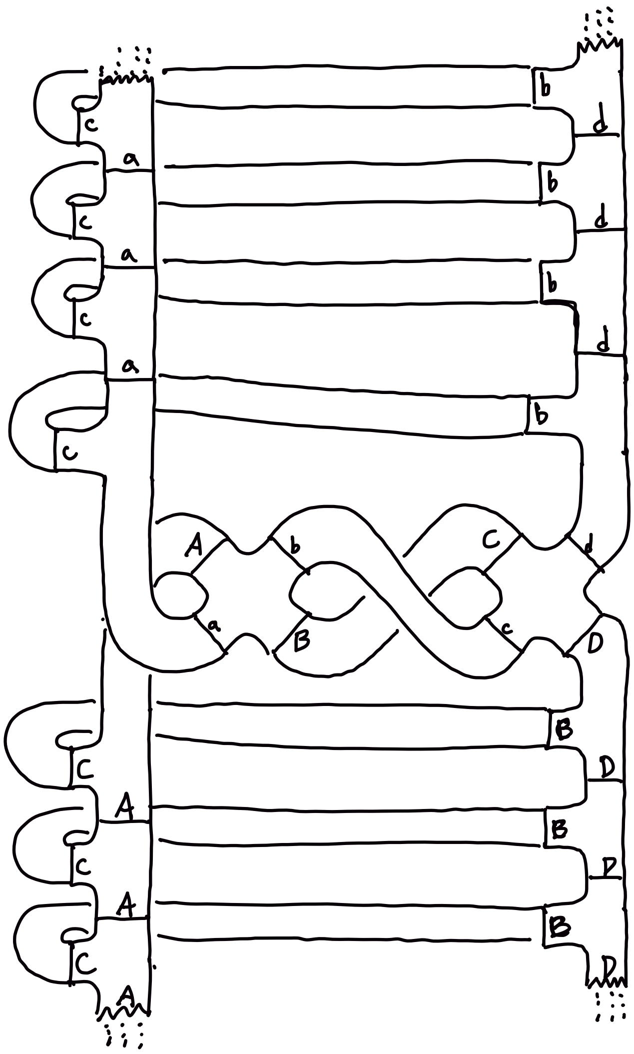


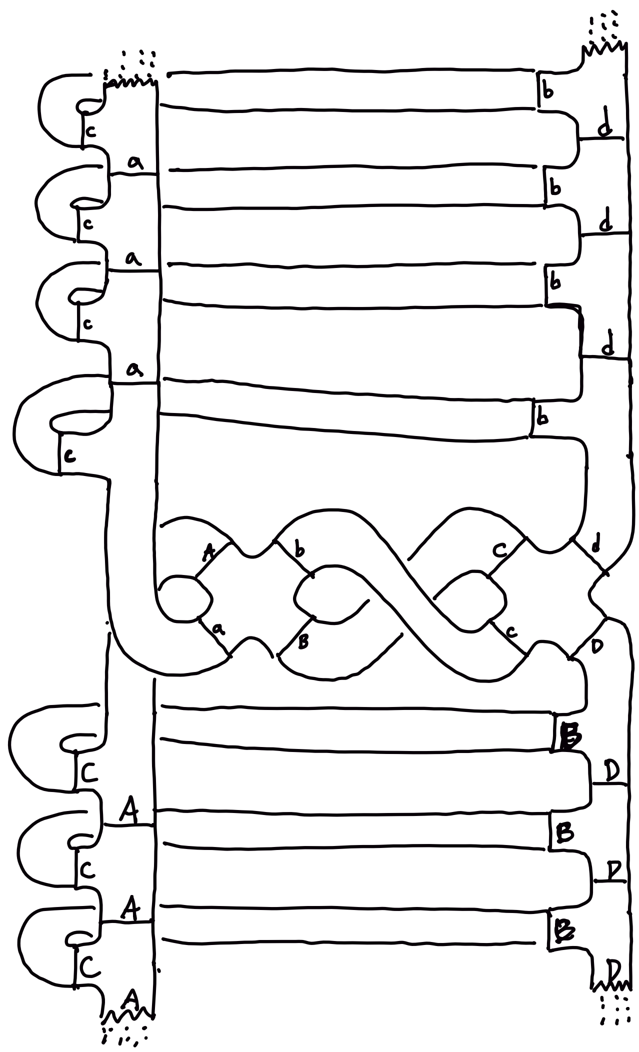


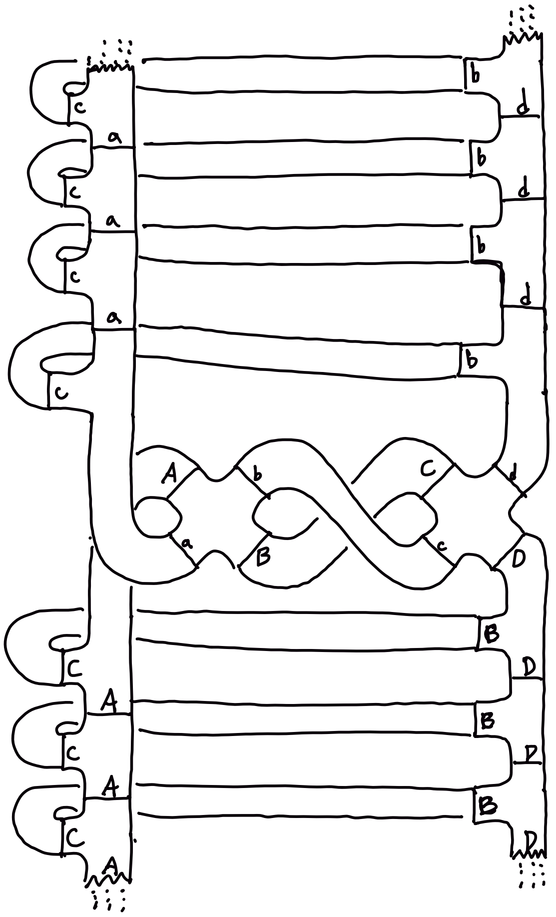




Blank copies of complicated surface for drawing on:







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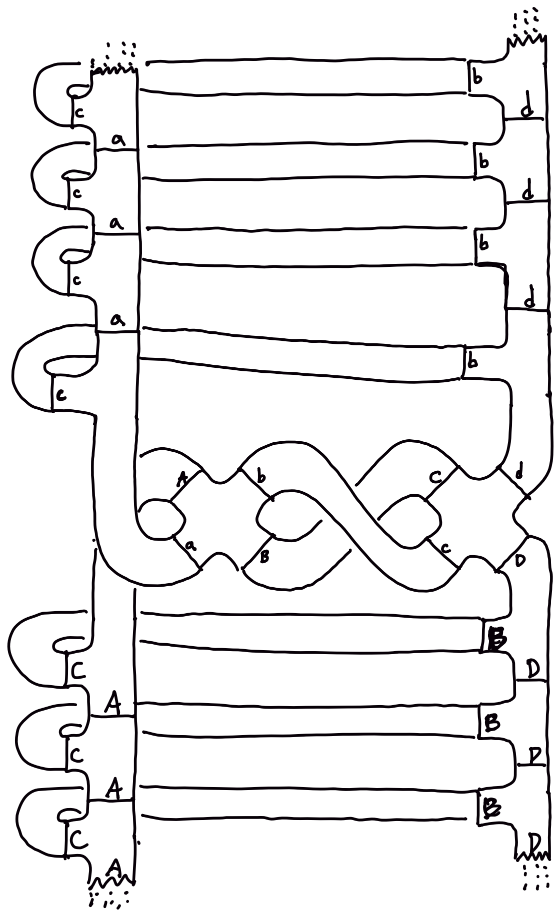

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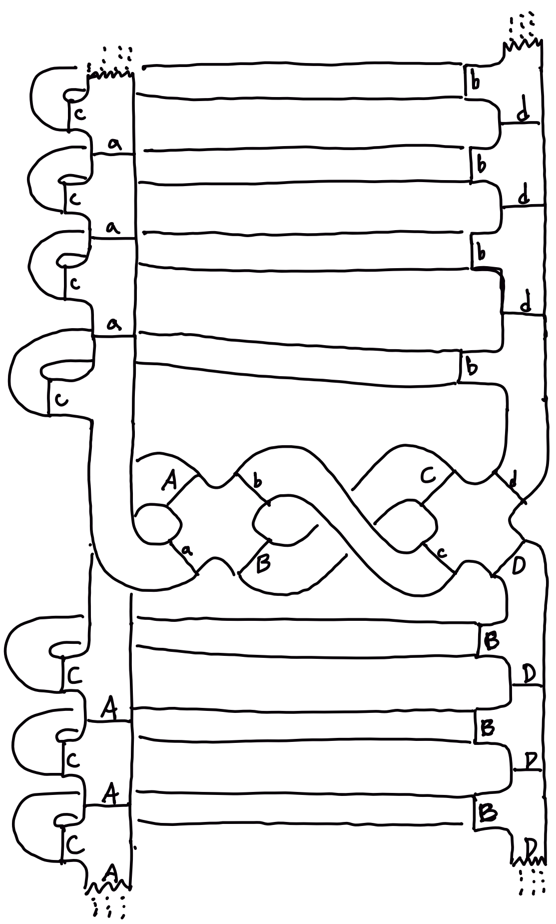

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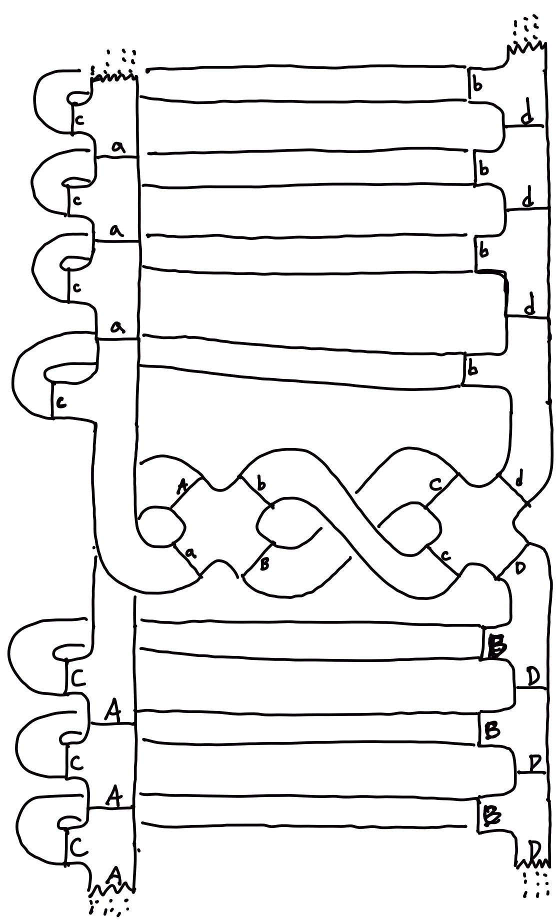

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