Immersed Curves and holomorphic triangles

jt. w/ Jesse Cohen

(I) Background

- Heegaard Floer homology (Ozsváth–Szabó)

$Y^3 \rightsquigarrow \widehat{HF}(Y)$

$W: Y_0 \to Y_1 \rightsquigarrow F_W : \widehat{HF}(Y_0) \to \widehat{HF}(Y_1)$

- Bordered Floer homology (Lipshitz–Ozsváth–Thurston)

$(Y, \partial Y \cong F)$

- $F \rightsquigarrow \mathcal{A}(F)$ dga. $\quad F = T^2$, $\underline{d=0}$.
- $(Y, \partial Y \cong F) \rightsquigarrow$ (1) $\widehat{CFD}(Y)$ type D

  (2) $\widehat{CFA}(Y)$ type A

- $\widehat{CF}(Y_0 \cup Y_1) \simeq \widehat{CFA}(Y_0) \boxtimes \widehat{CFD}(Y_1)$

$$\simeq \underbrace{\mathrm{Mor}_{\mathcal{A}}(\widehat{CFA}(Y_0), \widehat{CFA}(Y_1))}_{\mathrm{Mor}(Y_0, Y_1)}$$

Thm:

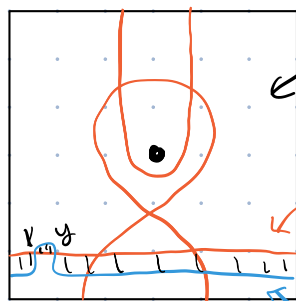
-
$$\begin{tikzcd}
\widehat{CF}(Y_0 \cup Y_1) \otimes \widehat{CF}(Y_1 \cup Y_2) \arrow[r] \arrow[d, "\simeq"'] & \widehat{CF}(Y_0 \cup Y_2) \arrow[d, "\simeq"] \\
\mathrm{Mor}(Y_0, Y_1) \otimes \mathrm{Mor}(Y_1, Y_2) \arrow[r, "\circ^{op}"'] & \mathrm{Mor}(Y_0, Y_2)
\end{tikzcd}$$

(Cohen)

Geometric Interpretation of Hanselman–Rasmussen–Watson when $\partial Y = T^2$

- $\widehat{CFD}(Y) / \widehat{CFA}(Y) \longmapsto \underbrace{\coprod S^1}_{\text{decorated immersed curves.}} \looparrowright \partial Y \setminus pt$ $\quad \theta_Y$

$\widehat{CF}(S^3_0(4_1))$



$\widehat{CFD}(S^3 \setminus 4_1)$

$\widehat{CFD}(S^1 \times D^2)$

$\partial x = y + y = 0$.

- $\widehat{CF}(-Y_0 \cup Y_1) \simeq \mathrm{Mor}_{\mathcal{A}}(Y_0, Y_1)$

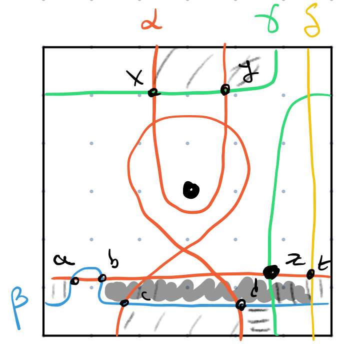
  $\qquad \mathbb{F}\langle \theta_{Y_0} \cap \theta_{Y_1} \rangle$ $\leftarrow$ ∂ counts bigons

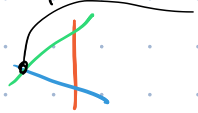
$\partial x = y + \dots$

Thm (Cohen-G):

$$\begin{tikzcd} Mor(Y_0,Y_1)\otimes Mor(Y_1,Y_2) \arrow[r,"\circ"] \arrow[d,"\simeq"] & Mor(Y_0,Y_2) \arrow[d,"\simeq"] \\ CF(\Theta_{Y_0},\Theta_{Y_1})\otimes CF(\Theta_{Y_1},\Theta_{Y_2}) \arrow[r,"m_2"'] & CF(\Theta_{Y_0},\Theta_{Y_2}) \end{tikzcd}$$

counts holomorphic triangles





unique intersection btwn β+γ.

x

$$\begin{array}{ccccc} S^3_0(4_1) & & S^3_1(4_1) & & \cancel{S^3_\infty(4_1)} \\ \widehat{CF}(\alpha,\beta) & \longrightarrow & \widehat{CF}(\alpha,\gamma) & \longrightarrow & \cancel{\widehat{CF}(\alpha,\delta)} \end{array}$$

$a \mapsto z$, b, $c \mapsto x$, $d \mapsto y$; $\cancel{t} \mapsto b$

surgery ~~exact~~ triangle

Thm:

$$\begin{tikzcd} \widehat{CF}(Y_0\cup Y_1)\otimes\widehat{CF}(Y_1\cup Y_2) \arrow[r] \arrow[d,"\simeq"] & \widehat{CF}(Y_0\cup Y_2) \arrow[d,"\simeq"] \\ Mor(Y_0,Y_1)\otimes Mor(Y_1,Y_2) \arrow[r,"\circ^{op}"'] & Mor(Y_0,Y_2) \end{tikzcd}$$

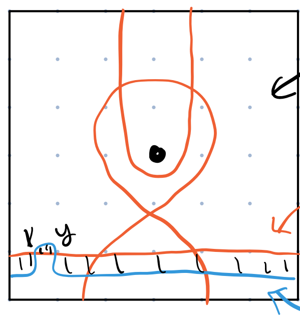
(Cohen)

Geometric Interpretation of Hanselman-Rasmussen-Watson when $\partial Y = T^2$

- $\widehat{CFD}(Y)/\widehat{CFA}(Y) \longmapsto \coprod S^1 \overset{\Theta_Y}{\longrightarrow} \partial Y$

decorated immersed curves.

$\widehat{CF}(S^3_0(4_1))$



$\widehat{CFD}(S^3\setminus 4_1)$

$\widehat{CFD}(S^1\times D^2)$

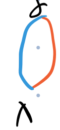
$\partial x = y + y = 0.$

- $\widehat{CF}(-Y_0\cup Y_1) \simeq Mor_{\mathcal{A}}(Y_0,Y_1)$

IS

$$\widehat{CF}(\Theta_{Y_0},\Theta_{Y_1}) = \mathbb{F}\langle \Theta_{Y_0}\cap\Theta_{Y_1}\rangle$$

2 counts bigons

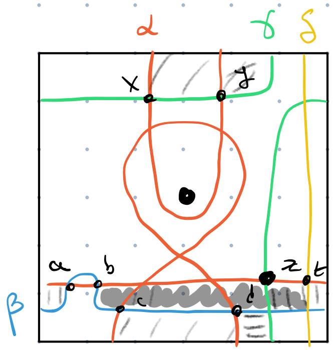


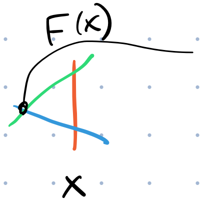
$\partial x = y + \dots$

Thm (Cohen–G):

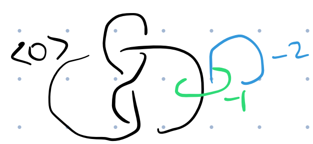
$$\begin{tikzcd} \mathrm{Mor}(Y_0,Y_1)\otimes \mathrm{Mor}(Y_1,Y_2) \arrow[r, "\circ"] \arrow[d, "\simeq"] & \mathrm{Mor}(Y_0,Y_2) \arrow[d, "\simeq"] \\ CF(\Theta_{Y_0},\Theta_{Y_1})\otimes CF(\Theta_{Y_1},\Theta_{Y_2}) \arrow[r, "m_2"] & CF(\Theta_{Y_0},\Theta_{Y_2}) \end{tikzcd}$$

counts holomorphic triangles



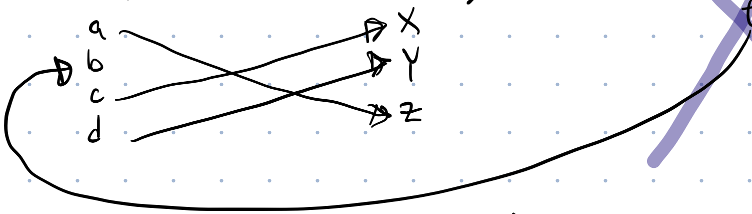


unique intersection btwn $\beta + \gamma$.



$S^3_0(4_1)$ $S^3_1(4_1)$ $S^3_\infty(4_1)$

$\widehat{CF}(\alpha,\beta) \longrightarrow \widehat{CF}(\alpha,\gamma) \longrightarrow \cancel{\widehat{CF}(\alpha,\delta)}$



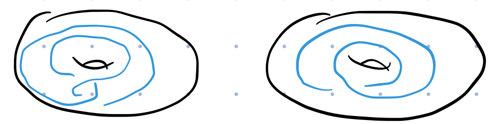
~~Surgery exact triangle~~

Knots: Chen / Chen–Hanselman

$S^1 \overset{P}{\hookrightarrow} S^1\times D^2$ a "(1,1) pattern"

$\exists$ curve α_P.

Ex



$K\subset S^3,\ P \rightsquigarrow P(K) := (S^3\setminus K)\cup(S^1\times D^2, P).$

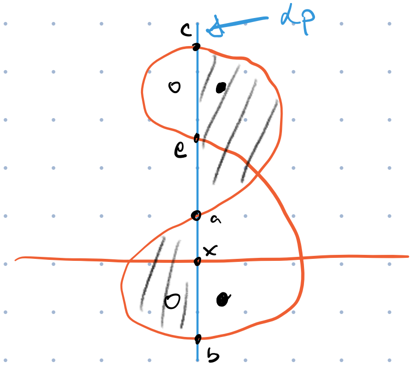
Thm (C/CH): If $\Theta_K \rightsquigarrow \widehat{CFD}(S^3\setminus K)$,

$\alpha_P \rightsquigarrow P$

$$CF^-(\alpha_P,\Theta_K) \xrightarrow{\simeq} CFA^-(P)\boxtimes CFD(S^3\setminus K)$$

is

$$CFK^-(P(K))$$



$$\begin{tikzcd} b & a\ \ x \arrow[l, "U"'] \arrow[d, "V"] \\ e \arrow[u, leftarrow, "V"'] & c \arrow[l, "U"] \end{tikzcd}$$

Thm (Cohen-Guth): Say $C : K_0 \to K_1$ concordance,

$P \rightsquigarrow \alpha_P$, $\Theta_{K_0}, \Theta_{K_1}$.

"satellite concordance"

$$\begin{tikzcd}
CF(\alpha_P, \Theta_0) \arrow[r, "\simeq"] \arrow[d, "M_2(-, F_{S^3\times I \setminus C})"'] & CFA^-(\alpha_P) \boxtimes \widehat{CFD}(S^3 \setminus K_0) \arrow[d, "\mathrm{id} \boxtimes F_{S^3\times I\setminus C}"] \\
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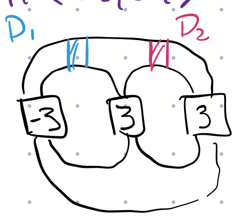
$F_{S^3\times I\setminus C} \in \mathrm{Mor}_A(S^3\setminus K_0, S^3\setminus K_1) \simeq$

$CF(\Theta_0, \Theta_1) \simeq \mathbb{F}\langle \Theta_0, \Theta_1 \rangle$

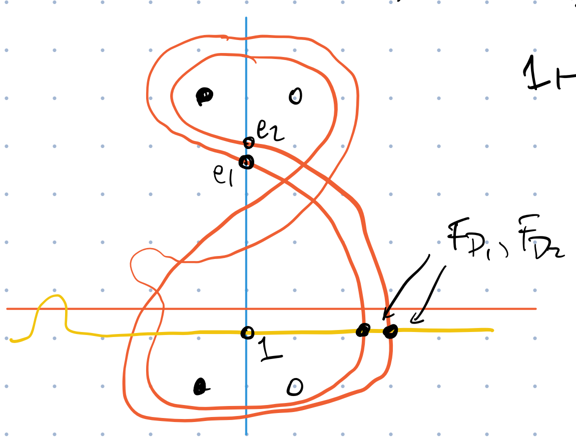
Ex: $K = P(-3,3,3)$
$P = \mathrm{Id}$



$F_{B^4\setminus D_1}, F_{B^4\setminus D_2} \in \mathrm{Mor}(S^3\setminus K, S^3\setminus K)$

$F_{D_1} + F_{D_2}$

$1 \mapsto e_1 + e_2 \neq 0$



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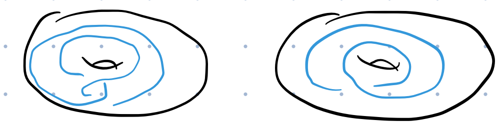

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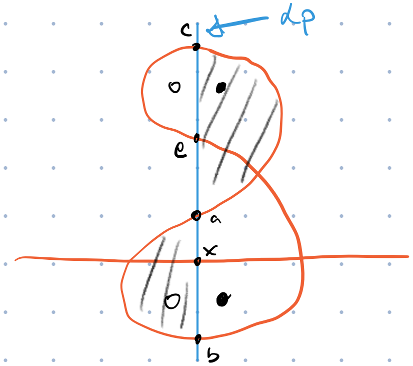
is
$CFK^-(P(K))$



$$\begin{tikzcd}
b \arrow[d, "V"'] & a \arrow[l, "U"'] \arrow[d, "V"] & x \\
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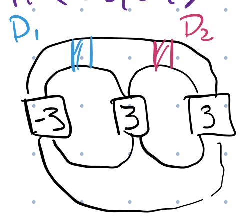

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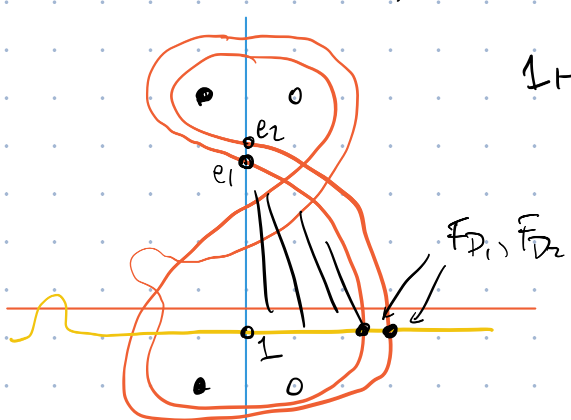
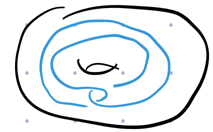
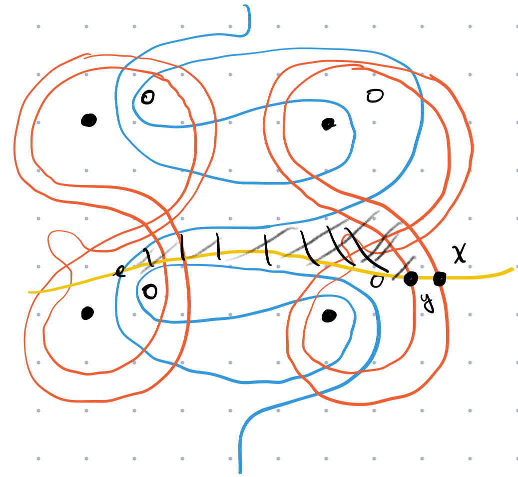


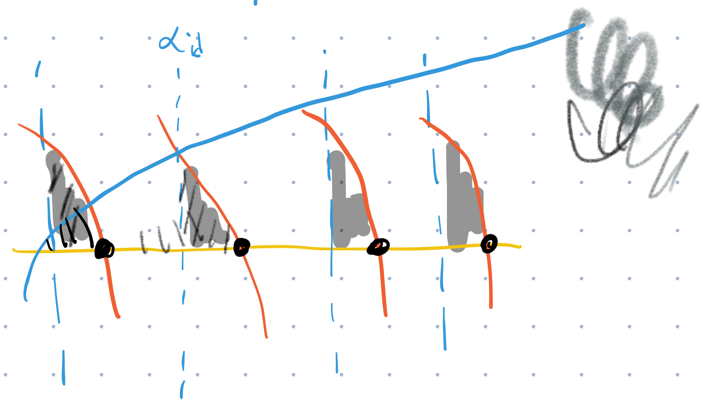
Change P: $P = Wh^+$



α_{Wh^+}:

$F_{Wh^+(D_1)} + F_{Wh^+(D_2)} = x + y$
$\neq 0$





Fact: $Wh(D), Wh(D')$ are TOP isotopic.

Thm (GHKP): If $F_D \neq F_{D'} \Rightarrow Wh(D), Wh(D')$ are exotic.

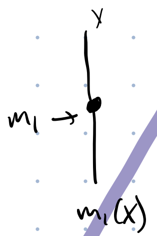
Idea of Proof:

Def: An A_∞-module M over A is v.s.? w/ maps

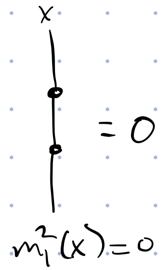
$$m_{k+1} : M \otimes A^{\otimes k} \longrightarrow M$$

satisfying A_∞-relations...

$m_1 : M \longrightarrow M$

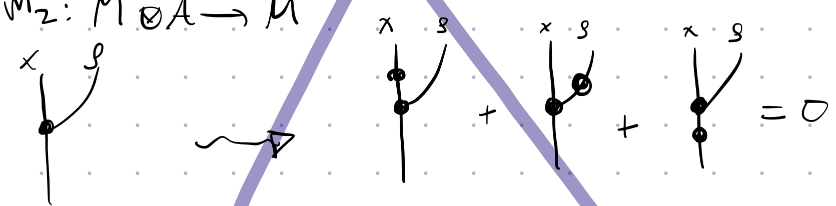


rel:



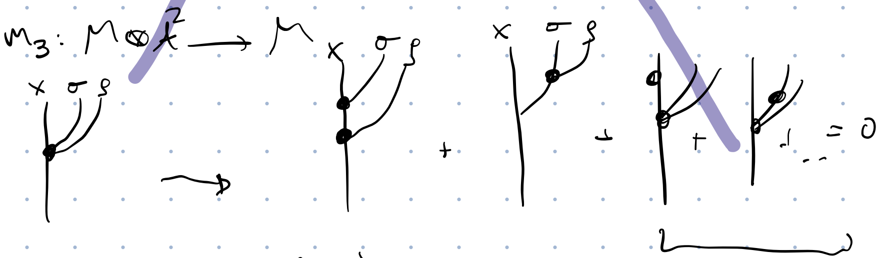
$\Rightarrow m_1$ is diff.

$m_2 : M \otimes A \longrightarrow M$



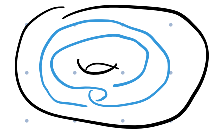
$$m_2(\partial x, \rho) + m_2(x, \partial \rho) = \partial m_2(x, \rho)$$

$m_3 : M \otimes A^{\otimes 2} \longrightarrow M$



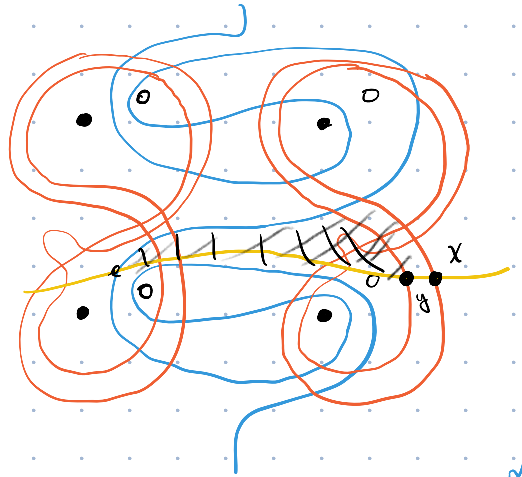
$\underbrace{(x \cdot \sigma) \cdot \rho + x \cdot (\sigma \cdot \rho)}_{\text{htpic via } m_3}$ $+ \; \partial \circ m_3 + m_3 \circ \partial^{\otimes} = 0$

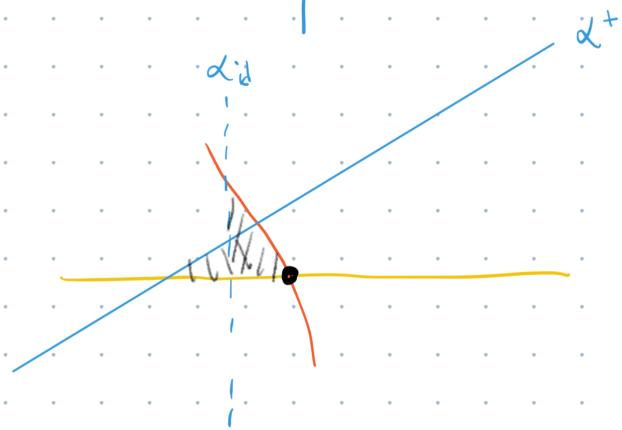
Change P: $P = Wh^+$



α_{Wh^+}:

$F_{Wh^+(D)} + F_{Wh^+(D_2)} = x + y$
$\neq 0$





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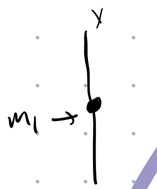

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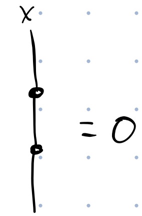
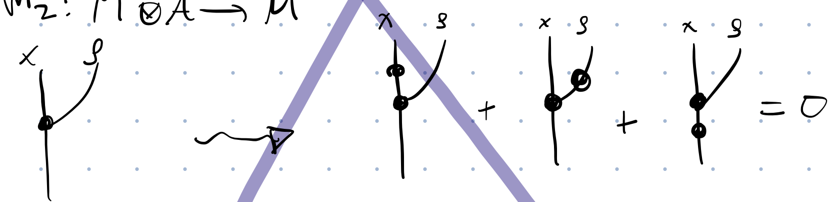


$m_1^2(x) = 0$

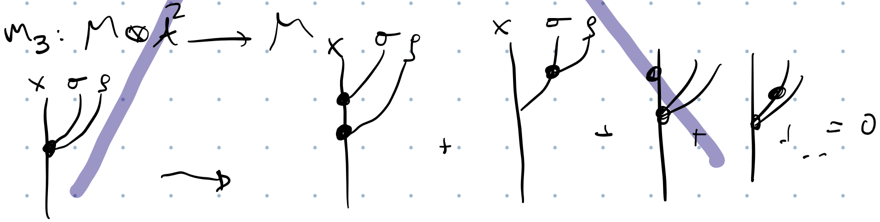
$\Rightarrow m_1$ is diff.

$m_2 : M \otimes A \to M$



$m_2(\partial x, \rho) + m_2(x, \partial \rho) = \partial m_2(x, \rho)$

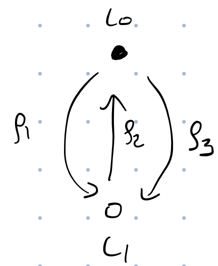
$m_3 : M \otimes A^2 \to M$



$$\underbrace{(x \cdot \sigma) \cdot \rho + x \cdot (\sigma \cdot \rho)}_{\text{htpic via } m_3} + \overbrace{\partial \cdot m_3 + m_3 \cdot \partial^{\otimes}}^{} = 0$$

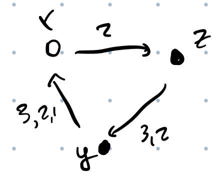
$\widehat{CFA}(Y)$ is A_∞-module over

$A(T^2) =$



$\rho_3 \cdot \rho_2 = 0$

$\rho_2 \cdot \rho_1 = 0$

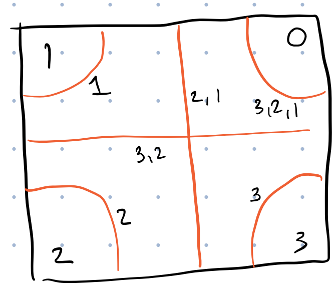


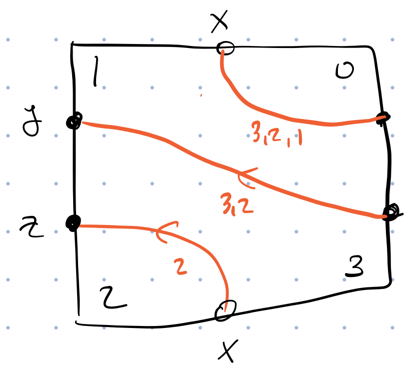
$\longleftrightarrow$

$m_2(x, \rho_2) = y$

$m_3(x, \rho_2, \rho_1) = z$

$m_2(x, \rho_3) = z$





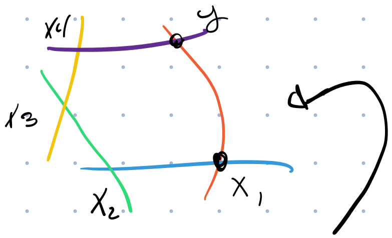
HRW: $\exists$ model for $\widehat{CFA}(Y)$ so its graph can realized by immersed curve in $T^2 \setminus pt$.

Def: The Fukaya category of $T^2 \setminus pt$ is A_∞-cat. consisting of

(1) Objects = (decorated) immersed curves in $T^2 \setminus pt$

(2) $\hom_{\mathcal{F}}(\theta_0, \theta_1) = \mathbb{F}\langle \theta_0 \cap \theta_1 \rangle$

(3) $\mu_k : \hom(\theta_0, \theta_1) \otimes \cdots \otimes \hom(\theta_{k-1}, \theta_k) \to \hom(\theta_0, \theta_k)$ given by counting holomorphic $k+1$ gons



$$m_4(x_1, x_2, x_3, x_4) = y.$$

HRW: ① $\widehat{CFA}(Y) \longleftrightarrow$ object of $\mathcal{F}$

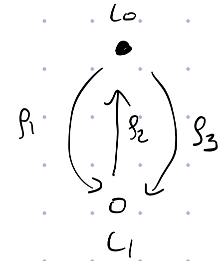
② $\widehat{CF}(Y_0 \cup Y_1) \longleftrightarrow$ morphisms of $\mathcal{F}$

③ Cobordisms (i.e. comp of CFA maps) $\longleftrightarrow$ composition in $\mathcal{F}$ $\underline{m_2}$.

⓪ $\mathcal{A}(T^2) \longleftrightarrow$?? (Auroux)

$\widehat{CFA}(Y)$ is A_∞-module over

$\mathcal{A}(T^2) =$



$/ \quad \rho_3 \cdot \rho_2 = 0$

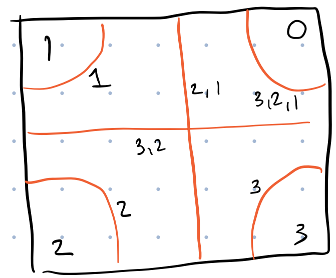
$\rho_2 \cdot \rho_1 = 0$

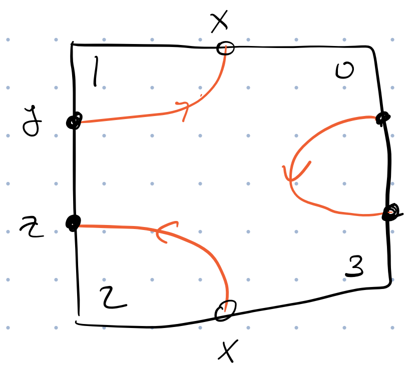
$\longleftrightarrow$

$m_2(x, \rho_2) = y$

$m_3(x, \rho_2, \rho_1) = z$

$m_2(x, \rho_3) = z$

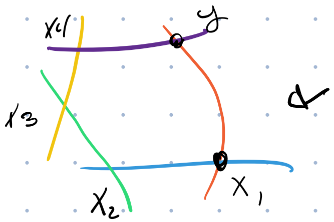




HRW: $\exists$ model for $\widehat{CFA}(Y)$ so its graph can realized by immersed curve in $T^2 \setminus pt$.

Def: The Fukaya category of $T^2 \setminus pt$ is A_∞-cat. consisting of

1. Objects = (decorated) immersed curves in $T^2 \setminus pt$
2. $\hom_{\mathcal{F}}(\theta_0, \theta_1) = \mathbb{F}\langle \theta_0 \cap \theta_1 \rangle$
3. $\mu_k : \hom(\theta_0,\theta_1) \otimes \cdots \otimes \hom(\theta_{k-1},\theta_k) \to \hom(\theta_0,\theta_k)$ given by counting holomorphic $k+1$ gons

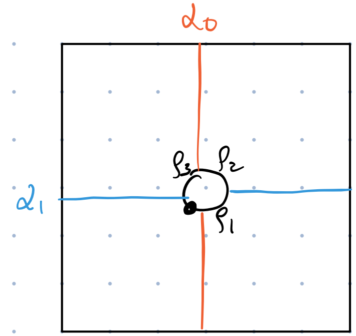


$$m_4(x_1, x_2, x_3, x_4) = y.$$

HRW:

1. $\widehat{CFA}(Y) \longleftrightarrow$ object of $\mathcal{F}$
2. $\widehat{CF}(Y_0 \cup Y_1) \longleftrightarrow$ morphisms of $\mathcal{F}$
3. Cobordisms (i.e. comp of CFA maps) $\longleftrightarrow$ composition in $\mathcal{F}$ m_2.

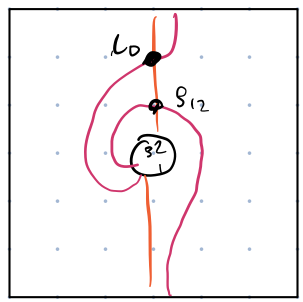
0. $\mathcal{A}(T^2) \longleftrightarrow$?? (Auroux)



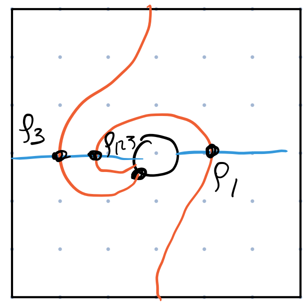
$$\mathrm{End}_{\mathcal{F}}(\alpha_0 \oplus \alpha_1) = \bigoplus_{i,j \in \{0,1\}} \hom(\alpha_i, \alpha_j)$$

has mult: $m_2 : \mathrm{End}^{\otimes 2} \to \mathrm{End}$.

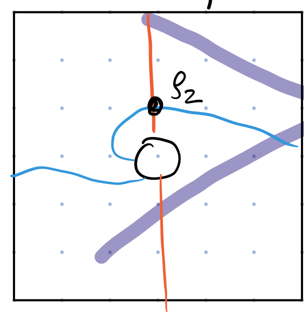
$\hom(\alpha_0, \alpha_0)$



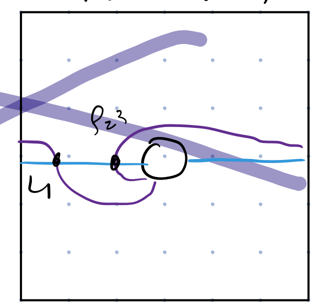
$\hom(\alpha_0, \alpha_1)$

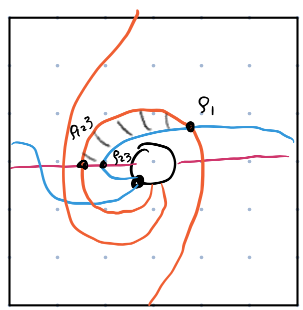


$\hom(\alpha_1, \alpha_0)$



$\hom(\alpha_1, \alpha_1)$





$$\rho_1 \cdot \rho_{23} = \rho_{123}.$$

$$\hom(\alpha_0,\alpha_1) \otimes \hom(\alpha_1,\alpha_1) \to \hom(\alpha_0,\alpha_1)$$

Thm (Auroux):

$$\mathcal{A}(T^2) \cong \mathrm{End}_{\mathcal{W}}(\alpha_0 \oplus \alpha_1)$$

• $\Upsilon : \mathcal{F}uk(T^2 \setminus pt) \longrightarrow \text{Mod-}End(\alpha_0\oplus\alpha_1)$

$$\theta \longmapsto \hom_F(\theta, \alpha_0\oplus\alpha_1)$$

$$f\in \hom(\theta_0,\theta_1) \longmapsto m_2(-,f) + \dots$$

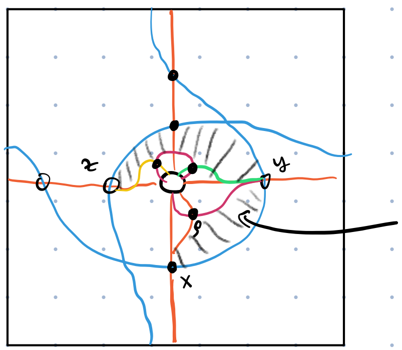
$$\hom_F(\theta_1, \alpha_0\oplus\alpha_1)\otimes End(\alpha_0\oplus\alpha_1)^{\otimes k} \xrightarrow[m_{k+1}]{} \hom_F(\theta, \alpha_0\oplus\alpha_1)$$

count polygons

Q: Say $\widehat{CFA}(Y) \rightsquigarrow \theta_Y$.

What is $\Upsilon(\theta_Y)$?

A: Exactly $\widehat{CFA}(Y)$.



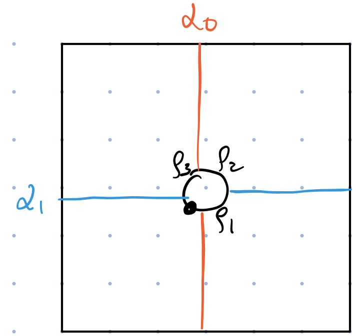
• $\in \iota_0\cdot\widehat{CFA}$

∘ $\in \iota_1\cdot CFA$

$m_2(x,\rho)=y$

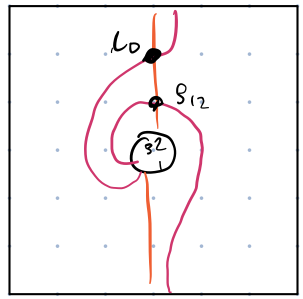
$m_3(y,\sigma,z)=z$

these correspond exactly to edges in graph defining $\widehat{CFA}$.

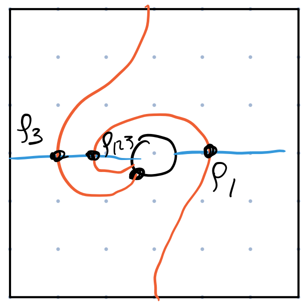


$$End_F(\alpha_0\oplus\alpha_1) = \bigoplus_{i,j\in\{0,1\}} \hom(\alpha_i,\alpha_j)$$

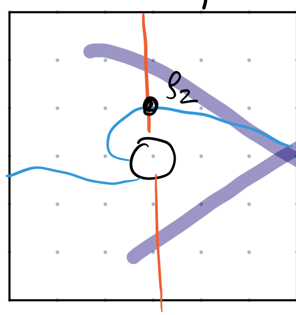
$\hom(\alpha_0,\alpha_0)$



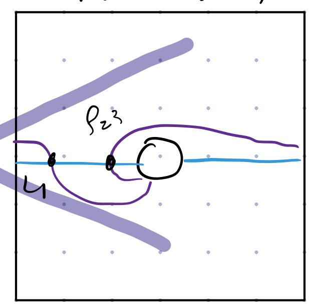
$\hom(\alpha_0,\alpha_1)$

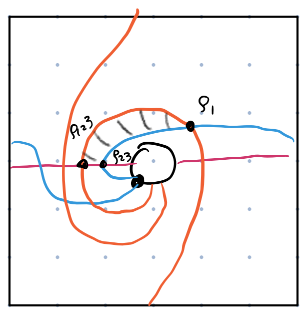


$\hom(\alpha_1,\alpha_0)$



$\hom(\alpha_1,\alpha_1)$





$\rho_1\cdot\rho_{23} = \rho_{123}$.

$\hom(\alpha_0,\alpha_1)\otimes\hom(\alpha_1,\alpha_1)\to\hom(\alpha_0,\alpha_1)$

Thm (Auroux):

$$\mathcal{A}(T^2)\cong End_W(\alpha_0\oplus\alpha_1)$$

(2) $\mathcal{Y} : \mathrm{Fuk}(T^2 \setminus pt) \longrightarrow \mathrm{Mod}\text{-}\mathrm{End}(\alpha_0 \oplus \alpha_1)$

$$\theta \longmapsto \hom_{\mathcal{F}}(\theta, \alpha_0 \oplus \alpha_1)$$

$$f \in \hom(\theta_0, \theta_1) \longmapsto m_2(-, f) + \dots$$

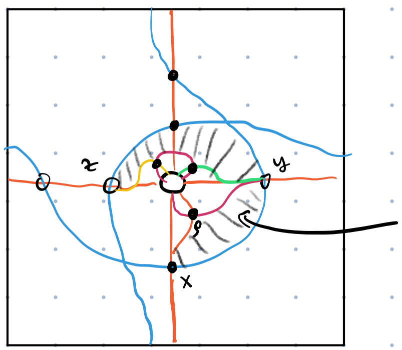
$$\hom_{\mathcal{F}}(\theta_1, \alpha_0 \oplus \alpha_1) \otimes \mathrm{End}(\alpha_0 \oplus \alpha_1)^{\otimes k} \xrightarrow[m_{k+1}]{} \hom_{\mathcal{F}}(\theta, \alpha_0 \oplus \alpha_1)$$

count polygons

Q: Say $\widehat{CFA}(Y) \mapsto \theta_Y$.

What is $\mathcal{Y}(\theta_Y)$?

A: Exactly $\widehat{CFA}(Y)$.



• $\in C_0 \cdot \widehat{CFA}$

○ $\in C_1 \cdot CFA$

$m_2(x, \rho) = y$

$m_3(y, \sigma, z) = z$

these correspond exactly to edges in graph defining $\widehat{CFA}$.

(3) In our case, $\mathcal{Y}$ is fully faithful embedding

$$\hom_{\mathcal{F}}(\theta_Y, \theta_X) \xrightarrow{\simeq} \hom_{\mathrm{End}(\alpha_0 \oplus \alpha_1)}(\mathcal{Y}(\theta_Y), \mathcal{Y}(\theta_X))$$

$$= \hom_{\mathcal{A}(T^2)}(CFA(Y), CFA(X))$$

$$= \widehat{CF}(-Y \cup X) \quad \square$$

respect composition