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IV.

Surfaces in 4-manifolds

What if $SW_X \equiv 0$?

$X = 2\mathbb{CP}^2$, $\quad H_2(2\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z} = \langle h_1, h_2\rangle$

Naive bd: $\quad g_X(d_1h_1, d_2h_2) = \frac{1}{2}(d_1-1)(d_1-2) + \frac{1}{2}(d_2-1)(d_2-2)$

$\qquad\qquad\qquad\qquad 0$ if $d_1=0 \qquad\qquad 0$ if $d_2=0$

In particular: $\quad g_X(3,0)=1$

$\mathbb{CP}^2 \qquad\qquad h_1$

but:

$-h_2 \quad \to \quad g_X(3,0)=0$

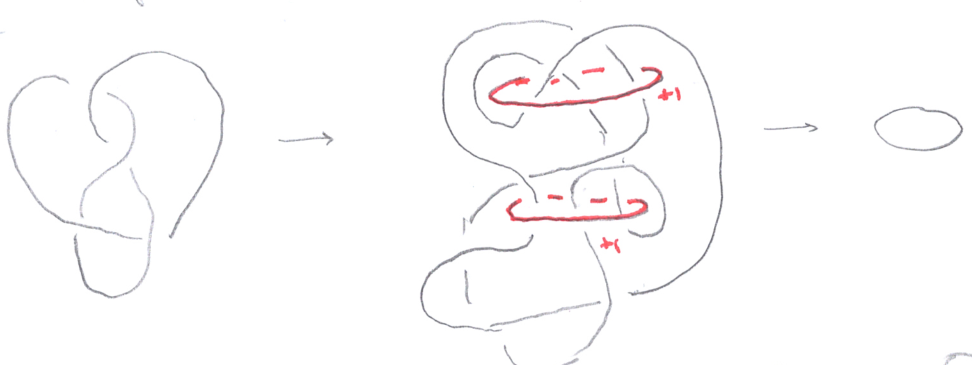
$y^2z = x^3 + x^2z$
nodal curve

Claim: $\quad g_X(3,1)=1$ $\quad \overset{+1}{\bigcirc} \subset 2\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$

Proof: If sphere $\to$ blow up 9 times $\to$ ... But $(3,1)$ char., so blow down $\to$ get X with $\sigma=-8$, complement spin $\to$ X is spin $\to$ ↯

Corollary: Figure-8 is not slice

Proof:



Shows an annulus in $2\mathbb{CP}^2 - 2D^4$ from Fig-8 $\to$ O

Cap off O. The annulus is in class $(3,1)$ —

Cannot cap off Fig-8

(Milnor-Fox would also work)

<u>Thm</u> (Bryan) $g_X(6,2) = 10$ (= 10+0 — as naive)

<u>Idea</u>: Take double branched cover → get spin manifold with extra $\mathbb{Z}_2$-action → SW contradiction (improve 10/8 with action)

Dai-Kang-Mallick-Park-S

<u>Corollary</u>: (2,1)-cable of Fig-8 is <u>not</u> slice.

(Miyazaki 1994: not ribbon)

<u>Proof</u>: Cable annulus ~ get $(F_8)_{(2,1)} \to T_{2,-9}$, bounds genus-9.

Genus of knots in $\mathbb{CP}^2$:

<u>Prop</u> (Marengon-Miller-A Ray-S)

$$g_{\mathbb{CP}^2}(T_{n,n-1}) \le \begin{cases} g_4(T_{n,n-1}) - \frac{n-2}{2} & n \equiv 0 \ (2) \\ g_4(T_{n,n-1}) - \frac{n-1}{2} & n \equiv 1 \ (2) \end{cases}$$

So improvement is large

<u>Proof</u>: Explicit construction of cob.

<u>Corollary</u>: For the class $(n,d) \in H_2(2\mathbb{CP}^2)$

Naive bound — $g_{2\mathbb{CP}^2} \ge \frac{n-3d}{2}$ $(n \equiv d \ (2))$

$\ge \frac{n-3d+1}{2}$ $(n \not\equiv d \ (2))$

<u>Proof</u>: Take $T_{n,n-1} \# -T_{n,n-1} \subset \partial(2\mathbb{CP}^2 - \text{disk})$

slice & slice in D^4 so have $g_{\mathbb{CP}^2}(T_{n,n-1})$

<u>Expectation</u>: · Naive bd works for $(3n, n)$ and (m,n)

$3n \ge m \ge n$

· Not outside of third region.

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Thm (AMPS) If $T_{p,q}$ is slice in $\mathbb{CP}^2 \Rightarrow T_{p,q} = T_{2,3}$.

Proof Sketch

Spheres in manifolds: Take $K3 = E(1) \# E(1)$

- $S \subset K3$ has negative square (adj. $\Rightarrow \leq 0, = 0$ })
- S^2 even (all squares are even)

Fact: $\exists$ sphere $S \subset K3$ with $S^2 = -n$ n even $2, \dots, 86$

no others.

Construction: Take DBC of $S^2 \times S^2$  get

$\downarrow$

$\mathbb{CP}^1$

singular. If we blow up + $\longrightarrow$

get smooth X with fibration, two

sing. fibers X 5 (−2)-curves

 $\to \mathbb{CP}^1$ shows that $X = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$

Get  21 (−2)-spheres $\longrightarrow$ −82 sphere

In monodromies: $(ab)^{12} = ((ab)^3)^4$

Another fibration: $((ab)^4)^3$ gives  $\times 3$ 22 spheres

$\downarrow$

−86

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$K \subset K3 - D^4$ slice?

Thm: If $u(K) \leq 21 \Rightarrow K$ is slice in $K3$

Proof (sketch)

BTW: · All knots are slice in $S^2 \times S^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (easy)

· How about in $2\mathbb{CP}^2$?