Seiberg-Witten theory

X smooth, closed, oriented; want $SW_X : H^2(X;\mathbb{Z}) \to \mathbb{Z}$

<u>Idea</u>: define PDE & count solution.

Assume $\pi_1(X) = 1$.

$K \in H^2(X,\mathbb{Z})$ gives rise to: $\cdot\ W^\pm \to X$ $U(2)$-bundles with
with $K \equiv w_2(X)\ (2)$ $\cdot\ \det W^+ \cong \det W^-\ (c_1 = K)$
$\cdot\ W^+ \otimes (W^-)^* \xrightarrow{\cong} TX \otimes \mathbb{C}$

The group behind: $Spin^c(4) = \{(A,B) \in U(2)\times U(2) \mid \det A = \det B\}$
with hom. $Spin^c(4) \xrightarrow{pr_\pm} U(2)$ gives $W^\pm$
$Spin^c(4) \longrightarrow SO(4)$ gives TX
$(A,B) \longmapsto (\underset{\mathbb{C}^2}{\overset{\cap}{v}} \longmapsto A v B^{-1})$

$Spin^c(4) \to SO(4)$ nontrivial S^1-bundle (the unique)

<u>Fact</u>: Riemann metric g on X gives Levi-Civita connection
(for vector fields X, Y, Z: covariant diff.: $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
& torsion free $\nabla_X Y - \nabla_Y X = [X,Y]$)

Choose connection A on $\det W^+ \to$ get covariant diff.
on W^+, W^- $W^+ \otimes T^*X \xrightarrow{c} W^-$ Clifford mult. then gives

$$\not{\partial}_A : \Gamma(W^+) \to \Gamma(W^-)$$

$\Lambda^2 \mathbb{R}^4 \cong Lie(SO(4)) = \{A \in M_4(\mathbb{R}) \mid A + A^T = 0\}$
$\omega_A(v,w) = g(v, Aw) \longleftarrow A$
$\| $
$\underset{+}{Lie(SO(3))} \oplus \underset{-}{Lie(SO(3))}$ (as $\widetilde{SO}(4) = SU(2)\times SU(2)$
$(q_+, q_-) \mapsto (x \mapsto q_+ x q_-^{-1})$

So for $A \longmapsto F_A$ curvature split as $F_A^+ + F_A^-$

(only in dim 4)

UGA
II

$\mathrm{Spin}^c(4) \longrightarrow SO(3)$

$(A,B) \longmapsto v \overset{V_+}{\longmapsto} AvA^{-1}$ $\to$ gives $\Lambda^+ X$

$\qquad\qquad BvB^{-1}$ $\qquad \Lambda^- X$

$\overset{V_-}{\mathrm{Im}\,\mathbb{H}} \subset \mathbb{H}$

$\overset{W_+}{\mathbb{H}} \longrightarrow \overset{V_+}{\mathrm{Im}\,\mathbb{H}}$ $\quad \mathrm{Spin}^c(4)$-equiv. $\rightsquigarrow$ gives map

$h \longmapsto -hih$ $\qquad \Gamma(W^+) \to \Gamma(\Lambda^+ X) = \Omega^+ X$

Equations: $\not{D}_A \Psi = 0$ $\quad$ for A, Ψ

$F_A^+ = i\sigma(\Psi)\ (+i\delta)$ ← perturbation for smoothness

<u>Fact</u>: • $\mathcal{G} = \mathrm{Map}(X, S^1)$ acts by $A \mapsto A + g\,dg$
$\qquad \Psi \mapsto g \cdot \Psi$

• Solutions (up to $\mathcal{G}$) $= \mathcal{M} \subset \{(A,\Psi)\}/_{\mathcal{G}} = \mathcal{B}$

• $\mathcal{M}$ is compact, smooth (for generic choices), orientable with $d = \frac{1}{4}(K^2 - 3\sigma - 2\chi)$
so $[\mathcal{M}] \in H_d(\mathcal{B}; \mathbb{Z})$

• $\mathcal{B} \sim \mathbb{CP}^\infty \longrightarrow$ get $SW_X(K) \in \mathbb{Z}$

<u>Blow-up formula</u>: relates SW_X & $SW_{X \# \overline{\mathbb{CP}^2}}$

$K \in H^2(X; \mathbb{Z}) \longrightarrow K \pm E \in H^2(X \# \overline{\mathbb{CP}^2}; \mathbb{Z})$
$\qquad E = PD(e)$

<u>Fact</u>: $SW_X(K) = \pm SW_{X \# \overline{\mathbb{CP}^2}}(K \pm E)$

<u>Idea</u>: Pull & analyze on $\overline{\mathbb{CP}^2}$ — unique reducible $(\Psi = 0)$

<u>Invariance statement</u>: If $b_2^+(X) > 0 \Rightarrow$ gen. g & δ gives SW

If $b_2^+(X) > 1 \Rightarrow$ these choices can be connected
$\rightsquigarrow$ invariance

$f: X_1 \to X_2$ diffeo

$\boxed{SW_{X_2}(K) = SW_{X_1}(f^* K)}$

Thm: $\Sigma \subset X$ with $\Sigma^2 \geq 0$ & $SW_X(k) \neq 0$
$\Rightarrow$ $K(\Sigma) + \Sigma^2 \leq 2g(\Sigma) - 2$

Can assume $\Sigma^2 = 0$ by repeated blow-up.

Idea of proof: $X = (\Sigma \times D^2) \cup_{\Sigma \times S^1} X - \nu(\Sigma)$

pick metric on Σ: constant curvature & area 1

on $\Sigma \times S^1$: length of $S^1 = 1$

$\Rightarrow$ area of $\Sigma \times S^1 = 1$ & scalar curvature $= 8\pi(1-g)$

$[-T, T]$ C

longer & longer neck

$$\Rightarrow \int_X s^2 \, dvol = T (8\pi)^2 (g-1)^2 + C$$

Chern-Weil on $\Sigma \times \{t\} \times \{\theta\}$

for a connection A
$$\int_{\Sigma \times t \times \theta} |F_A|_\Sigma|^2 \, dvol \geq 4\pi^2 \langle c_1(s), \Sigma \rangle^2$$

$$\Rightarrow \int_X |F_A|^2 \, dvol \geq 4\pi^2 T \langle c_1(s), [\Sigma] \rangle^2$$

Fact: (A, Φ) solution of SW, then

$$\boxed{\frac{1}{4}\int_X |F_A|^2 + \int_X |\nabla_A \Phi|^2 + \frac{1}{4}\int \left(|\Phi|^2 + \frac{s}{2}\right)^2 - \int_X \frac{s^2}{16} \underbrace{- \frac{1}{4}\int F_A \wedge F_A}_{\text{topological}: \ -\pi^2 c_1^2(s)[X]} = 0}$$

from Weitzenbock 94-95

$$4\pi^2 T \langle c_1, \Sigma \rangle^2 \leq \int_X |F_A|^2 \leq -4\pi^2 c_1^2(s)[X] + \frac{1}{4}\int_X s^2 =$$
$$= -4\pi^2 c_1^2(s)[X] + \frac{T}{4}(8\pi^2)(g-1)^2 + \frac{C}{4}$$

divide by $4\pi^2 T$ & $T \to \infty$ gives

$\langle c_1(s), \Sigma \rangle^2 \leq (2g-2)^2$, concluding.

The Weitzenböck formula for D_A:

$$D_A^* D_A \Psi = \nabla_A^* \nabla_A \Psi + \frac{s}{4}\Psi - \frac{1}{2} F_A^+ \cdot \Psi \quad / \langle \,, \Psi \rangle$$

& apply when (A, Ψ) is a solution
(so $D_A \Psi = 0$ & $F_A^+ = \sigma(\Psi \Psi)$), getting
the identity

Lecture 2. potential exercises

Fact: Q_X indef. $\Rightarrow \exists$ class α with $\alpha^2 = 0$

- Show that if $\sigma(Q) = 0 \Rightarrow Q = n\langle 1\rangle \oplus n\langle -1\rangle$ or $m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Show that if Q is indefinite & odd $\Rightarrow Q = n\langle 1\rangle \oplus m\langle -1\rangle$
- K char. $\Rightarrow K^2 \equiv \sigma(Q)$ (8)
- $Spin(3) = SU(2)$; find map to $SO(3)$
  $Spin(4) = SU(2) \times SU(2)$ find map to $SO(4)$
  $Spin^c(3) = U(2) \longrightarrow SO(3)$ unique non-triv. S^1-ble
  $Spin^c(4) = SU(2) \times SU(2) \times S^1 / \pm(1,1,1) \longrightarrow SO(4)$ unique non-triv. S^1-ble.