The genus function

Geometry of four-manifolds.

X^4 closed, oriented, smooth 4-mfold.

Fact: $\forall \alpha \in H_2(X;\mathbb{Z})$ $\exists$ $f: \Sigma_g \hookrightarrow X$ smooth emb. s.t. $f_*[\Sigma_g] = \alpha$. Indeed: $L \underset{\sigma}{\to} X$ with $PD(c_1) = \alpha$, σ gen. section, $\sigma^{-1}(0) = \Sigma$.

Def: $g_X : H_2(X;\mathbb{Z}) \to \mathbb{N}$

$$\alpha \longmapsto \min\{g(\Sigma) \mid [\Sigma] = \alpha\}$$

genus-function

Fact: In many cases g_X distinguishes between homeomorphic four-manifolds.

X complex surface & $C \subset X$ cpx curve (connected)

Adjunction equality: $\underbrace{2g(\Sigma) - 2}_{\text{genus}} = \underbrace{C \cdot C - \langle c_1(X), [C] \rangle}_{\text{homological}}$

Indeed, $TX|_C = TC \oplus \nu C$, $c_1(TC) = \chi(C) = 2-2g$, $c_1(\nu C) = C \cdot C$ + Whitney formula.

In fact, if (X, J) almost-complex, $\Sigma \subset X$ J-holomorphic ($JT\Sigma = T\Sigma$), same argument works.

E.g. (X, ω) symplectic, $\Sigma \subset X$ symplectic surface ($\omega|_{T\Sigma}$ sympl)

$\Rightarrow \exists J$ comp. a.c. & Σ J-hol.

So genus of sympl. curves: determined by homology.

Example: In $\mathbb{CP}^2$: $H_2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z} = \langle h \rangle$

dh represented by d lines, after smoothing smooth sympl. surface Σ_d

$$\boxed{g(\Sigma_d) = \tfrac{1}{2}(d-1)(d-2)}$$

Observation: • $g_X(-\alpha) = g_X(\alpha)$ & $g_{-X} = g_X$

Example: • $S^2 \times S^2$: $H_2(S^2\times S^2;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z} = \langle [S^2\times\{pt.\}], [pt.\times S^2] \rangle$

x, y

$ax+by$ cpx if $a,b \geq 0$

genus: $\boxed{(a-1)(b-1)}$ if $a,b \geq 1$; if $b=0 \Rightarrow g=0$

If $a>0$, $b<0$, flip orientation of $S^2\times S^2$

• $S^2 \tilde{\times} S^2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, $H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2};\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z} = \langle h, e \rangle$

$\downarrow$ S^2 $\quad ah+be$, $a>0$ & $a \geq b$

blow up

[blow-up: $\tau = \{ ((x,y),[u:v]) \in \mathbb{C}^2\times\mathbb{CP}^1 \mid xv = yu \}$

$\swarrow$ $\mathbb{C}^2$ blow-up $\quad \searrow$ $\mathbb{CP}^1$ tautological bundle]

Determine genus!

Recall: Seiberg-Witten invariant

$SW_X: H^2(X;\mathbb{Z}) \to \mathbb{Z}$ diffeom. inv. $(b_2^+>1)$

Def: $K \in H^2(X;\mathbb{Z})$ is a basic class if $SW_X(K) \neq 0$

Thm: (Adjunction inequality) X smooth, $b_2^+>1$, $\Sigma \hookrightarrow X$ with $\Sigma\cdot\Sigma \geq 0$, $[\Sigma]\neq 0$. If K is a basic class

$$\Longrightarrow \quad |K([\Sigma])| + \Sigma\cdot\Sigma \leq 2g(\Sigma) - 2$$

Thm (Taubes) X symplectic & $b_2^+ > 1 \Rightarrow$
$SW_X(c_1(X,\omega)) = \pm 1$

Thm (Ozsváth - Szabó) (X,ω) symplectic, $C \subset (X,\omega)$ symplectic curve, $\Sigma \subset X$ smooth curve.
If $[C] = [\Sigma] \Rightarrow g(C) \le g(\Sigma)$.
Symplectic Thom conjecture (C is genus minimizing)

Corollary: We know the genus function for $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$,
$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, $S^2 \times S^2$.
And of course S^4.

→ (A)

Refine: · For X as before, $X^\circ = X - \text{int}\, D^4$ with $\partial X^\circ = S^3$

$$G_X : H_2(X;\mathbb{Z}) \times \mathcal{K} \longrightarrow \mathbb{N}$$

isotopy classes of knots in S^3

$$(\alpha, K) \longmapsto \min \{ g(F) \mid [F, \partial F] = \alpha,\ \partial F = K \}$$

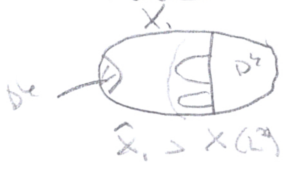
· Can be extended to links as well.

Def: · For X let S_X denote the set of slice links, i.e. $S_X = \{ L \subset S^3 \mid \text{comp's of } L \text{ bd disjoint disks} \}$

· An exotic D^4 X is small if $X \hookrightarrow D^4$ and large otherwise (motivated by small / large exotic $\mathbb{R}^4$'s)

Thm (C-S): If X is small exotic $D^4 \Rightarrow S_X = S_{D^4}$

Proof: Follows from: $X_1 \subset X_2 \Rightarrow S_{X_1} \subseteq S_{X_2}$.
Trace embedding lemma: L slice in $X \iff X(L^*) \hookrightarrow \hat{X}$
0-fr. trace



but then $X(L^*) \subset X_1 \subset X_2 \subset \hat{X}_2 \Rightarrow L$ slice in X_2.

So we cannot detect them easily

Rem: $\exists$ small exotic $D^4 \iff$ smooth 4-d Schönflies conj. fails

(that $S^3 \subset S^4$ has standard ball on one of its sides)

Thm (C-S): If X is large exotic & geom. simply conn. (that is, $\exists$ handle decomp with no 1-h) $\Rightarrow S_X \supsetneq S_{D^4}$

[Can be detected by genus: if X contractible & $\partial X \neq S^3 \Rightarrow$ X is not gsc]

Ⓐ Consider $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ (the orientation reversed is not symp. anywhere) $H_2 = \mathbb{Z}^3 = \langle h, e_1, e_2 \rangle$ $\quad \alpha = ah + b_1e_1 + b_2e_2$

Observation: reflection to (-1)-sphere can be induced by diffeo (cpx conjugation)

Fact: $g_X(\alpha) = g_X(f_*\alpha)$ $\forall X$ & $\forall f \in \mathrm{Diff}^+(X)$ this.

If $a \geq b_1 + b_2$ — repr. by symplectic

If $c = a - b_1 - b_2 < 0$ $\longrightarrow$ reflect to $h + e_1 + e_2$ (-1)-sphere — get $a - 2c, b_1 - 2c, b_2 - 2c$, make them ≥ 0 & repeat; it will stop.

Corollary: If $\alpha \in H_2(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}; \mathbb{Z})$ has $\alpha^2 \geq 0 \Rightarrow$ $g(\alpha)$ can be computed. If $\alpha^2 = 0 \Rightarrow g(\alpha) = 0$.

Fact: $\exists$ symplectic $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ X with $\Sigma_2 \subset X$ symplectic & $\Sigma_2^2 = 0 \rightsquigarrow X$ is exotic.

UGA I

LECTURE 1 potential exercises:

- Show that cpx smooth curve $C \subset \mathbb{CP}^2$ of degree d has genus $\frac{1}{2}(d-1)(d-2)$

- Review blow-up & show that $\alpha \in H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}; \mathbb{Z})$ with $\alpha^2 = 0$ can be represented by a sphere (Same in $S^2 \times S^2$)

- Same in $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$.

-  $\subset S^3 = \partial(\mathbb{CP}^2 - \text{disk})$ is slice in $\mathbb{CP}^2 - \text{disk}$ (indeed, H-slice, i.e. disk is 0 in H_2)

- Trace embedding lemma: $K \subset S^3$ is slice in $D^4 \iff X_K = D^4 \cup$ 2-handle along K, fr$=0$ $\hookrightarrow S^4$